

# ON $\mu$ -COMPATIBLE METRICS AND MEASURABLE SENSITIVITY

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**ABSTRACT.** We introduce the notion of W-measurable sensitivity, which extends and strictly implies canonical measurable sensitivity, the measure-theoretic version of sensitive dependence on initial conditions. This notion also implies pairwise sensitivity with respect to a large class of metrics. We show that finite measure-preserving ergodic dynamical systems must be either W-measurably sensitive, or isomorphic to an ergodic isometry on a compact metric space.

## 1. INTRODUCTION

The notion of sensitive dependence on initial conditions is an extensively studied isomorphism invariant of topological dynamical systems on compact metric spaces ([GW93], [AAB96]). In [JKL<sup>+</sup>08], the authors define two measure-theoretic versions of sensitive dependence, measurable sensitivity and strong measurable sensitivity, and show that, unlike their traditional topologically-dependent counterpart, both of these properties carry up to measurable-theoretic isomorphism. James et. al. introduce these notions for nonsingular transformations and show that measurable sensitivity is implied by double ergodicity (a property equivalent to weak mixing in the finite measure-preserving case) and strong measurable sensitivity is implied by light mixing.

In this paper, we introduce W-measurable sensitivity, a notion that is *a priori* stronger than measurable sensitivity and implies it straightforwardly. We use this newly defined property, together with properties of  $\mu$ -compatible metrics (see below), to formulate a complete classification of all finite measure-preserving ergodic transformations on standard spaces as being either W-measurably sensitive or isomorphic to an ergodic isometry on a compact metric space (a Kronecker transformation). In the course of this proof, we also show that W-measurable sensitivity is in fact equivalent to measurable sensitivity for conservative and ergodic transformations.

In addition, we show (see Appendix B) that the notion of W-measurable sensitivity is closely related to *pairwise sensitivity*, a notion introduced in [CJ05] for finite measure-preserving transformations. In their paper, Cadre and Jacob show that weakly mixing transformations always exhibit pairwise sensitivity, and also any ergodic transformation satisfying a certain entropy condition. Our results imply that any ergodic transformation that

is not measurably isomorphic to a Kronecker transformation will exhibit pairwise sensitivity with respect to any  $\mu$ -compatible metric (in addition to W-measurable sensitivity).

Throughout the paper, the ordered quadruple  $(X, \mathcal{S}(X), \mu, T)$  denotes the standard Lebesgue space  $X$  with a nonnegative, finite or  $\sigma$ -finite, nonatomic measure  $\mu$ , with  $\mathcal{S}(X)$  the collection of  $\mu$ -measurable subsets of  $X$ . Furthermore, we assume  $T$  to be a measurable nonsingular endomorphism of  $X$ , i.e.  $\mu(A) = 0$  if and only if  $\mu(T^{-1}(A)) = 0$  for all  $A \in \mathcal{S}(X)$  (see [Sil08]). We often require that  $T$  is measure-preserving or that the measure space is actually finite.

In the course of this paper, we require two unusual assumptions. First of all, we require our measure spaces to have cardinality no larger than that of the real line (see Appendix A; this assumption can be avoided by varying our definitions slightly). Secondly, we are only able to carry out the proof for transformations that are *forward measurable*: we assume that whenever a set  $A$  is measurable, so is  $T(A)$  (See Section 4).

A metric  $d$  on  $(X, \mathcal{S}(X), \mu)$  is said to be  $\mu$ -compatible if  $\mu$  assigns positive (nonzero) measure to all nonempty, open  $d$ -balls in  $X$ . If  $d$  a  $\mu$ -compatible metric on  $(X, \mathcal{S}(X), \mu)$ , then  $X$  is separable under  $d$  [JKL<sup>+</sup>08]. Note that this implies that any nonempty set that is open under  $d$  is a countable union of sets of positive measure, and so has positive measure as well. All  $d$ -closed sets are therefore measurable, etc.

The plan of the paper is as follows. In Section 2 we define W-measurable sensitivity and show it can be equivalently expressed in two additional ways using properties of  $\mu$ -compatible metrics defined on nonsingular dynamical systems. In Section 3, we use 1-Lipshitz metrics and show how such metrics can be constructed from any  $\mu$ -compatible metric defined on a probability space, such that all balls under the new metrics are all measurable. In Section 4, we provide a sufficient condition under which the newly constructed 1-Lipshitz metric is in fact  $\mu$ -compatible. Section 5 illustrates the main connection between 1-Lipshitz metrics and W-measurable sensitivity, namely that a conservative and ergodic nonsingular dynamical system is W-measurably sensitive if and only if all dynamical systems  $(X', \mu', T')$  measurably isomorphic to it admit no  $\mu'$ -compatible 1-Lipshitz metrics. A corollary of this fact is that for ergodic finite measure-preserving transformations, W-measurable sensitivity is equivalent to measurable sensitivity as defined in [JKL<sup>+</sup>08]. Finally, in Section 6 we prove our main result, which classifies all ergodic, measure-preserving transformations on finite measure spaces as being either W-measurably sensitive, or isomorphic to a Kronecker transformation.

In Appendix A we discuss the invariance of W-measurable sensitivity under measurable isomorphism, as well as the technical assumptions necessary for it to hold. Appendix B elaborates on the relationship between our results and the notion of pairwise sensitivity as introduced in [CJ05].

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## 2. W-MEASURABLE SENSITIVITY

We start by recalling the definition of measurable sensitivity.

**Definition 2.1.** [JKL<sup>+</sup>08] *A nonsingular dynamical system  $(X, \mathcal{S}(X), \mu, T)$  is said to be measurably sensitive if for every isomorphic dynamical system  $(X_1, \mathcal{S}(X_1), \mu_1, T_1)$  and any  $\mu_1$ -compatible metric  $d$  on  $X_1$ , then there exists  $\delta > 0$  such that for  $x \in X_1$  and all  $\varepsilon > 0$  there exists  $n \in \mathbb{N}$  such that*

$$\mu_1\{y \in B_\varepsilon(x) : d(T_1^n(x), T_1^n(y)) > \delta\} > 0.$$

We now introduce the definition that we shall be using extensively.

**Definition 2.2.** *For a  $\mu$ -compatible metric  $d$ , a nonsingular dynamical system  $(X, \mu, T)$  is W-measurably sensitive with respect to  $d$  if there is a  $\delta > 0$  such that for every  $x \in X$ ,*

$$\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta$$

*for almost every  $y \in X$ . The dynamical system is said to be W-measurably sensitive if the above definition holds true for any  $\mu$ -compatible metric  $d$ .*

**Remark.** *For example, similarly to [JKL<sup>+</sup>08] it can be shown that a doubly ergodic nonsingular transformation is W-measurably sensitive. Double ergodicity is a condition equivalent to weak mixing in the finite measure-preserving case [Fur81].*

**Remark.** *On the other hand, it is easy to see that if a measure space  $(X, \mu)$  has atoms, no transformation on it can exhibit W-measurable sensitivity with respect to any metric. Indeed, for any  $x \in X$ , and any  $\delta$ , the set of points  $y$  such that  $\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta$  cannot include  $x$ . So this set cannot have full measure if  $\mu(\{x\}) > 0$ .*

*The same is not true about measurable sensitivity. For this reason, throughout this paper we have to assume that our measure space is non-atomic. This assumption is justified because most of the results of this paper concern ergodic transformations, and ergodic transformations on spaces with points of positive measure are all rotations on a discrete set.*

**Remark.** *A very important example of a dynamical system which is not W-measurably sensitive is a Kronecker transformation, i.e. an ergodic isometry on an interval of finite length (with the Lebesgue measure and the usual metric). This transformation is not W-measurably sensitive with respect to the usual metric because it is an isometry.*

We note that, under our mild additional assumption on the cardinality of the dynamical systems, the property of W-measurable sensitivity is preserved under measurable isomorphisms. These issues are discussed in Appendix A, but they are not used until Section 5.

Let us make a rough comparison of the two definitions above. Definition 2.2 states that for some  $\delta > 0$ , every  $x \in X$  satisfies the condition that for a set of *full measure*  $Y \subset X$ , for every  $y \in Y$ , there are *infinitely many* values of  $n$  for which  $d(T^n x, T^n y) > \delta$ . On the other hand, measurable sensitivity as defined in [JKL<sup>+</sup>08] only requires that there is a  $\delta > 0$  for which every  $x \in X$  satisfies the condition that, for every  $\varepsilon > 0$ , there is a set of *positive measure*  $Y \subset B_\varepsilon(x)$  such that for every  $y \in Y$  there is *one* value of  $n$  for which there is a set of positive measure  $Y \subset B_\varepsilon(x)$  and a single  $n$ , depending only on  $Y$ , such that  $d(T^n x, T^n y) > \delta$ . So, it should not be surprising that W-measurable sensitivity implies measurable sensitivity (the reader may want to skip to the first part of the proof of Proposition 5.2 where this argument is made precise).

In fact, we show that the two notions are equivalent for ergodic dynamical systems. We first show in Proposition 2.1 that for a transformation to be W-measurably sensitive, it is sufficient for each  $y \in Y$  to have *one* value of  $n$  that satisfies  $d(T^n x, T^n y) > \delta$ . The remainder of the equivalence follows from the results in the following sections, culminating with Proposition 5.2.

**Proposition 2.1.** *Let  $(X, \mu, T)$  be a nonsingular dynamical system, and  $d$  be a  $\mu$ -compatible metric. The following are equivalent:*

- (1) *The system is W-measurably sensitive with respect to  $d$ .*
- (2) *There is a  $\delta > 0$  such that, given any  $x \in X$ , for almost every  $y \in X$ , there exists  $n$  such that  $d(T^n x, T^n y) > \delta$ .*
- (3) *There is a  $\delta > 0$  such that for every  $x \in X$ ,*

$$\mu \{y \in X \mid \forall n \geq 0, d(T^n x, T^n y) < \delta\} = 0.$$

*Proof.* (2)  $\Rightarrow$  (1). Suppose that there is a  $\delta > 0$  such that, given any  $x \in X$ , for almost every  $y \in X$ , there exists  $n$  such that  $d(T^n x, T^n y) > \delta$ . Let us fix some  $x_0 \in X$ . For every natural number  $N$  define a set  $Y_N$  by:

$$Y_N = \{y \in X \mid \exists n > N, d(T^n x_0, T^n y) > \delta\}.$$

We now prove that for all  $N$ , the set  $Y_N$  has full measure. Consider the point  $T^N x_0$ . Using our assumption, for almost every  $y \in X$ , there exists  $n$  such that  $d(T^n(T^N x_0), T^n y) > \delta$ . In other words, the set

$$Z_N = \{y \in X \mid \exists n > 0, d(T^{N+n} x_0, T^n y) > \delta\}$$

has full measure.

Notice that  $Y_N = T^{-N}(Z_N)$ . Since  $T$  is a nonsingular transformation,  $Y_N$  must also have full measure.

Finally, let  $Y = \bigcap_{N=0}^{\infty} Y_N$ . Clearly,  $Y$  has full measure. Furthermore, for every  $y \in Y$ , there are infinitely many values of  $n$  such that  $d(T^n x_0, T^n y) >$

$\delta$ . So

$$\limsup_{n \rightarrow \infty} d(T^n x_0, T^n y) \geq \delta$$

for almost all  $y \in X$ . Since  $x_0$  was an arbitrary point in  $X$ , the system  $(X, \mu, T)$  is W-measurably sensitive with respect to  $d$ .

(1)  $\Rightarrow$  (2). If a system is W-measurably sensitive with respect to  $d$ , then there is a  $\delta > 0$  such that given any  $x \in X$ ,

$$\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta$$

for almost every  $y \in X$ . This clearly implies that there is some  $n$  for which  $d(T^n x, T^n y) > \delta$ .

(2)  $\Leftrightarrow$  (3). For any  $x \in X$  and  $\delta > 0$ , let us denote by  $B$  the set  $\{y \in X \mid \forall n \geq 0, d(T^n x, T^n y) < \delta\}$ . If condition (2) is satisfied at  $x$  for some  $\delta$ , then  $B$  is contained in the complement of a set of full measure. So  $\mu(B) = 0$  and condition (3) is satisfied.

Conversely, if condition (3) is satisfied at  $x$  for some  $\delta$ , then  $B$  has measure zero. So in particular, the set  $\{y \in X \mid \forall n \geq 0, d(T^n x, T^n y) < \delta/2\}$  has measure zero. Therefore, for almost every  $y \in X$ , there is some  $n$  for which  $d(T^n x, T^n y) > \delta/2$ , and condition (2) is satisfied.  $\square$

### 3. CONSTRUCTING 1-LIPSHITZ METRICS

We shall use the term *1-Lipshitz metrics (with respect to  $T$ )* to denote metrics that satisfy the inequality  $d(Tx, Ty) \leq d(x, y)$  for all  $x$  and  $y$ .

First, we provide a way to construct a 1-Lipshitz metric from any other metric.

**Definition 3.1.** Let  $(X, \mu, T)$  be a dynamical system, and  $d$  be a  $\mu$ -compatible metric on  $X$ . For any two points  $x, y \in X$ , we define

$$d_T(x, y) = \min \left( \sup_{n \geq 0} d(T^n x, T^n y), 1 \right).$$

We shall use the notation  $B_{T_\delta}(x)$  for the set  $\{y \mid d_T(x, y) < \delta\}$ .

**Lemma 3.1.**  $d_T$  is a metric on  $X$ . Moreover, it is a 1-Lipshitz metric.

*Proof.* Clearly, since  $d$  is a metric, we must have that  $d_T(x, y) = 0$  if and only if  $x$  and  $y$  are the same point and also  $d_T(x, y) = d_T(y, x)$  for all  $x, y \in X$ .

Now, we need to prove the triangle inequality. Let  $x, y, z \in X$  and let  $f(x, y) = \sup_{n \geq 0} d(T^n x, T^n y)$ . Then, by the triangle inequality for the metric  $d$ ,

$$f(x, z) = \sup_{n \geq 0} d(T^n x, T^n z) \leq \sup_{n \geq 0} (d(T^n x, T^n y) + d(T^n y, T^n z))$$

However, we have that

$$\sup_{n \geq 0} (d(T^n x, T^n y) + d(T^n y, T^n z)) \leq \sup_{n \geq 0} d(T^n x, T^n y) + \sup_{n \geq 0} d(T^n y, T^n z)$$

Therefore,  $f(x, z) \leq f(x, y) + f(y, z)$ . It follows that  $d_T(x, z) \leq d_T(x, y) + d_T(y, z)$ . So  $d_T$  is indeed a metric.

Finally, let us check that our metric is 1-Lipshitz:

$$\begin{aligned} f(Tx, Ty) &= \sup_{n \geq 0} d(T^n(Tx), T^n(Ty)) = \sup_{n \geq 1} d(T^n x, T^n y) \\ &\leq \sup_{n \geq 0} d(T^n x, T^n y) = f(x, y). \end{aligned}$$

So, if  $f(x, y) < 1$ , then  $d_T(x, y) \geq d_T(Tx, Ty)$ . Clearly, if  $f(x, y) \geq 1$ , then  $d_T(x, y) = 1 \geq d_T(Tx, Ty)$ , and so  $d_T$  is 1-Lipshitz.  $\square$

We also have the following result:

**Lemma 3.2.** *Let  $(X, \mu, T)$  be a dynamical system. If  $d$  is a  $\mu$ -compatible metric and  $T$  is nonsingular then all the balls under the metric  $d_T$  will be measurable.*

*Proof.* If  $r > 1$ , then  $B_{T_r}(x) = X$ , which has positive measure since it contains the entire space. So, let's assume that  $r \leq 1$ . Then,

$$B_{T_r}(x) = \{y \in X \mid \sup_{n \geq 0} d(T^n x, T^n y) < r\} = \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq 0} T^{-n} \left( B_{r - \frac{1}{m}}(T^n x) \right)$$

Since  $d$  is a  $\mu$ -compatible metric and  $T$  is a nonsingular transformation, we have that  $T^{-n} \left( B_{r - \frac{1}{m}}(T^n x) \right)$  is measurable for all  $n$  and  $m$ . So  $B_{T_r}(x)$  must be measurable.  $\square$

**Remark.** *In general, even if the metric  $d$  is  $\mu$ -compatible, the metric  $d_T$  may not be  $\mu$ -compatible as its balls may have measure 0. Consequently, there is no guarantee that the measure space is separable under the topology determined by  $d_T$ , and it is conceivable that there are  $d_T$ -open sets (not expressible as a countable union of balls) which are not measurable.*

*For example, let  $I$  be the unit interval,  $\lambda$  be the Lebesgue measure, and  $d$  be the usual metric. Let  $T : I \rightarrow I$  be the doubling map  $Tx = 2x \pmod{1}$ . Note that  $d$  is a  $\lambda$ -compatible metric.*

*The metric  $d_T$ , however, is not  $\lambda$ -compatible. Indeed, for any  $x \notin \mathbb{Q}$ , and any  $\varepsilon > 0$ , there will be an  $n$  such that  $d(0, T^n x) > 1 - \varepsilon$ . So, since  $T(0) = 0$ , we have*

$$\sup_{n \geq 0} d(T^n(0), T^n y) = \sup_{n \geq 0} d(0, T^n y) = 1.$$

*In other words, for any  $0 < \delta < 1$ , the  $\delta$  ball around 0 in the  $d_T$  metric may contain only rational points. So,  $\lambda(B_{T_\delta}(0)) = 0$ , and  $d_T$  is not  $\mu$ -compatible.*

*Note that in this example, the transformation  $T$  will turn out to be  $W$ -measurably sensitive. In fact, since it's mixing, it is strongly measurably sensitive (see [JKL<sup>+</sup>08]). On the other hand, we will see that whenever the 1-Lipshitz metric  $d_T$  is  $\mu$ -compatible, the corresponding transformation  $T$  is not  $W$ -measurably sensitive. We will also see a sort of a converse to this statement in Proposition 5.1.*

4. CONDITIONS FOR 1-LIPSHITZ METRICS TO BE  $\mu$ -COMPATIBLE

Now, we provide a sufficient condition for the 1-Lipshitz metric  $d_T$  to be  $\mu$ -compatible given that the transformation  $T$  is ergodic. First, we shall prove a technical lemma:

**Lemma 4.1.** *Suppose  $d$  is a 1-Lipshitz metric on  $X$  with respect to a transformation  $T$ . Then, for any integer  $m > 0$  and any real number  $r > 0$ ,*

$$B_r(x) \subset T^{-m}(B_r(T^m x)).$$

*Proof.* Suppose  $y \in B_r(x)$ . Then  $d(x, y) < r$ . Since  $d$  is 1-Lipshitz, we must have  $d(T^m x, T^m y) < r$  for all positive integers  $m$ . So  $T^m y \in B_r(T^m x)$  and therefore  $y \in T^{-m}(B_r(T^m x))$ .  $\square$

From now on through the end of this paper, we will also need the following additional assumption:

*For all dynamical systems  $(X, \mu, T)$  we consider henceforth, we assume that  $T$  is **forward measurable**, i.e. that for all measurable sets  $A$ ,  $T(A)$  is also measurable.*

**Remark.** *Many dynamical systems satisfy this condition. In particular, any dynamical system which is locally invertible (i.e., systems such that  $X$  can be split into countably many components  $X_1, X_2, \dots$  such that  $T$  restricted to a map from  $X_i$  to  $T(X_i)$  has a measurable inverse for all  $i$ , see [Aar97, p.7]) has this property. See also Walters [Wal69].*

*Though this assumption is critical to our proof of Lemma 4.2 (without it, we have no way of showing that the set  $U_r$  is measurable, and therefore cannot use the fact that the system is ergodic for this set), it is not clear to us whether this assumption is actually necessary. We do not use this assumption explicitly anywhere else in our paper, though most results will depend on this Lemma.*

Now, we are ready to state the sufficient condition the 1-Lipshitz metric  $d_T$  to be  $\mu$ -compatible which we will use in the next section.

**Lemma 4.2.** *Let  $(X, \mu, T)$  be a conservative and ergodic nonsingular dynamical system such that  $T$  is forward measurable. Let  $d$  be a  $\mu$ -compatible metric on  $X$ . Suppose further that for any real number  $\delta > 0$ , there is an  $x \in X$  such that  $\mu(B_{T_\delta}(x)) > 0$ . Then, there is a set  $S$  of zero  $\mu$ -measure such that  $d_T$  is a  $\mu$ -compatible metric on the measure space  $X \setminus S$ . (We use  $\mu$  to represent both the original measure  $\mu$  and its restriction to  $X \setminus S$ .)*

*Proof.* Let  $r > 0$  and let  $U_r = \{x \in X \mid \mu(B_{T_r}(x)) > 0\}$ . We wish to show that  $U_r$  is measurable and has full measure. To do so, we will construct a set of full measure that is contained in  $U_r$ .

By our assumption, there is an  $x \in X$  such that  $\mu(B_{T_{r/2}}(x)) > 0$ . Let  $M = B_{T_{r/2}}(x)$ . By the triangle inequality, if  $y \in M$  then  $B_{T_r}(x) \supset M$ , and therefore  $y \in U_r$ . Furthermore, by Lemma 4.1, for any positive integer

$m$ , we have  $T^{-m}(B_r(T^m y)) \supset B_r(y) \supset M$  and therefore the first set in this containment must have positive measure. Since  $T$  is nonsingular, this implies that the set  $B_r(T^m y)$  has positive measure, and therefore  $T^m y \in U_r$  for all  $m$ .

So the set  $N = \bigcup_{m=0}^{\infty} T^m(M)$  is a subset of  $U_r$ . Since we assumed that  $T$  is forward measurable,  $N$  is measurable. Furthermore, it's clear that  $T^{-1}(N) \supset N$ . As  $T$  is conservative, we must have  $T^{-1}(N) = N$  (up to sets of measure zero). Since  $T$  is ergodic, and  $N$  does not have measure zero (as it contains  $M$ ), this implies that  $N$  has full measure (that is, the complement has zero measure). Since our measure is complete,  $U_r$  must be measurable and have full measure as well.

Now, the set  $U = \bigcap_{n=1}^{\infty} U_{1/n}$  also has full measure. Clearly, for any  $x \in U$  and any  $r > 0$ , the ball  $B_{T_r}(x)$  must have positive measure. So we set  $S = X \setminus U$  and we are done.  $\square$

## 5. MEASURABLE SENSITIVITY AND 1-LIPSHITZ METRICS

In this section, we prove and discuss the key result of our paper, which will provide the technical foundation for all our conclusions:

**Proposition 5.1.** *Let  $(X, \mu, T)$  be a conservative and ergodic nonsingular dynamical system.  $T$  is  $W$ -measurably sensitive if and only if all measurably isomorphic dynamical systems  $(X', \mu', T')$  admit no  $\mu'$ -compatible metrics that are 1-Lipshitz. In other words,  $(X, \mu, T)$  is  $W$ -measurably sensitive if and only if for all measurably isomorphic dynamical systems  $(X', \mu', T')$  and all  $\mu'$ -compatible metrics  $d'$  on  $X'$ , there are points  $x, y \in X'$  such that  $d'(Tx, Ty) > d'(x, y)$ .*

*Proof. Forward direction.* First of all, let us note that if a dynamical system  $(X', \mu', T')$  admits a  $\mu'$ -compatible 1-Lipshitz metric  $d'$ , then this system could not be  $W$ -measurably sensitive. Indeed, for any  $x$ , and any  $\delta > 0$ , the ball  $B'_\delta(x)$  has positive measure and for all points  $y \in B'_\delta(x)$  and all integers  $n$ ,  $d'(T^n x, T^n y) \leq d'(x, y) < \delta$ , as the metric is 1-Lipshitz.

Now, we use the invariance of  $W$ -measurable sensitivity under measurable isomorphism, which is proved in Appendix A. Suppose a dynamical system  $(X, \mu, T)$  is  $W$ -measurably sensitive. Then, according to Proposition A.3, any measurably isomorphic system  $(X', \mu', T')$  will also be  $W$ -measurably sensitive, and therefore will not admit a  $\mu'$ -compatible 1-Lipshitz metric  $d'$ .

*Backward direction.* Now, suppose  $(X, \mu, T)$  is *not*  $W$ -measurably sensitive. We need to find a measurably isomorphic dynamical system  $(X_1, \mu_1, T_1)$  and a  $\mu_1$ -compatible metric  $d_1$  that satisfies  $d_1(x, y) \geq d_1(Tx, Ty)$ .

By Proposition 2.1, since  $(X, \mu, T)$  is not  $W$ -measurably sensitive, there is some  $\mu$ -compatible metric  $d$  such that for each  $\delta > 0$ , there is a point  $x_\delta \in X$  and a set  $Y_\delta \subset X$  of positive measure that satisfies

$$\sup_{n \geq 0} d(T^n x_\delta, T^n y) < \frac{\delta}{2} \text{ for all } y \in Y_\delta.$$

Let us construct the 1-Lipshitz metric  $d_T$  as before. Clearly,  $Y_\delta \subset B_{T_\delta}(x_\delta)$ . So, for every  $\delta$ , we have found an  $x_\delta$  such that  $\mu(B_{T_\delta}(x_\delta)) > 0$ . Now, use Lemma 4.2 to construct a set  $S$  of measure 0 such that  $d_T$  is  $\mu$ -compatible on  $X \setminus S$ .

Since  $T$  is nonsingular, the set  $S_1 = \bigcup_{n \geq 0} T^{-n}S$  also has measure 0. We set  $X_1 = X \setminus S_1$ ,  $T_1$  to be  $T$  restricted to  $X_1$  (it is well-defined since, clearly,  $T(X_1) \subset X_1$ ), and  $\mu_1$  to be  $\mu$  restricted to  $X_1$ . Clearly, the system  $(X', \mu', T')$  is isomorphic to  $(X, \mu, T)$ . Furthermore,  $d_T$  is a 1-Lipshitz  $\mu_1$ -compatible metric on  $(X_1, \mu_1, T_1)$ .  $\square$

**Remark.** Note that, under the conditions of Proposition 5.1, it is possible that the system  $(X, \mu, T)$  is not  $W$ -measurably sensitive, but does not itself admit any  $\mu$ -compatible metric  $d$  such that for all points  $x, y \in X$ ,  $d(Tx, Ty) \leq d(x, y)$ .

For example, consider the dynamical system  $(I, \lambda, T)$  where  $I$  is the unit interval and  $\lambda$  is the Lebesgue measure. Let  $\alpha$  be a fixed irrational number between 0 and 1. For any  $x \in I$ , we define:

$$T(x) = \begin{cases} x & \text{if } x = n \cdot \alpha + m \text{ for some } n, m \in \mathbb{Z} \\ x + \alpha \pmod{1} & \text{otherwise.} \end{cases}$$

This system is ergodic and not measurably sensitive as it is measurably isomorphic to a rotation. However, there is no  $\lambda$ -compatible 1-Lipshitz metric on  $I$ .

Indeed, suppose that there is a  $\lambda$ -compatible metric  $d$  such that  $d(Tx, Ty) \leq d(x, y)$  for all  $x, y \in I$ . Let  $B$  be a ball of radius  $\alpha/2$  around 0. Since  $d$  is  $\lambda$ -compatible,  $B$  must have positive measure. Furthermore, since  $T(0) = 0$ , for any point  $b \in B$ , we must have  $d(T(b), 0) \leq d(b, 0) < \alpha/2$  and, therefore,  $T(b) \in B$ . So  $T$  maps a set of positive measure into itself. This is impossible for a transformation measurably isomorphic to an irrational rotation.

We conclude that it is in fact necessary to consider isomorphic dynamical system in Proposition 5.1.

As a first application of Proposition 5.1, we show the following proposition.

**Proposition 5.2.** *If a dynamical system is  $W$ -measurably sensitive, then it is measurable sensitive. If a dynamical system is conservative ergodic and measurably sensitive, then it is  $W$ -measurably sensitive.*

*Proof.* First, we show that  $W$ -measurable sensitivity implies measurable sensitivity. Suppose  $(X, \mu, T)$  is a  $W$ -measurably sensitive dynamical system. Then, by Proposition A.3, any measurably isomorphic dynamical system  $(X_1, \mu_1, T_1)$  is also  $W$ -measurably sensitive. So, for any  $\mu_1$ -compatible metric  $d_1$  on  $X_1$ , there is a  $\delta > 0$  such that for all  $x \in X_1$ , we have  $\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta$  for almost all  $y \in X_1$ .

In particular, we must certainly have

$$\mu_1\{y \in B_\varepsilon(x) : \exists n \text{ s.t. } d(T_1^n(x), T_1^n(y)) > \delta\} = \mu_1(B_\varepsilon(x)) > 0.$$

But the set from the above inequality is a countable union of the form

$$\bigcup_{n=0}^{\infty} \{y \in B_{\varepsilon}(x) : d(T_1^n(x), T_1^n(y)) > \delta\}.$$

Since the union has positive measure, there must be an  $n$  for which the set  $\{y \in B_{\varepsilon}(x) : d(T_1^n(x), T_1^n(y)) > \delta\}$  also has positive measure. Since this is true for all systems  $(X', \mu', T')$  measurably isomorphic to  $(X, \mu, T)$ ,  $(X, \mu, T)$  is measurably sensitive by Definition 2.1

Now, to show the other direction, suppose  $(X, \mu, T)$  is a conservative ergodic dynamical system that is not W-measurably sensitive. Then, by Proposition 5.1, there is a measurably isomorphic dynamical system  $(X_1, \mu_1, T_1)$  and a  $\mu_1$ -compatible metric  $d_1$  on  $X_1$  that is 1-Lipshitz. For all  $\delta > 0$ , we just pick any  $\varepsilon < \delta$ , and then for any  $x \in X_1$  and all integers  $n$  we will have

$$\{y \in B_{\varepsilon}(x) : d(T_1^n(x), T_1^n(y)) > \delta\} = \emptyset.$$

So neither  $(X_1, \mu_1, T_1)$  nor  $(X, \mu, T)$  can be measurably sensitive by Definition 2.1.  $\square$

**Remark.** (1). Note that the assumption that the dynamical system is ergodic is crucial to the above statement. For example, as we mentioned in section 2, no transformation can be W-measurably sensitive on a space with points of positive measure. Nonetheless, there are (non-ergodic) transformations on such spaces which are measurably sensitive according to the definition in [JKL<sup>+</sup>08].

(2) There is a delicate technical point in the proof of Proposition 5.2. The standard definition of a dynamical system on a Lebesgue space does not require the underlying space to have the same cardinality as the real line (there may be a measure zero set of large cardinality). So, to claim that  $(X, \mu, T)$  is measurably sensitive, in the context of Lebesgue spaces, we are technically required to consider measurably isomorphic systems  $(X', \mu', T')$  for which  $X'$  may have arbitrary cardinality (though  $X$  will still have reasonable cardinality). For this reason, we will carefully word the statement of Proposition A.3 so that it does not put any restrictions on the cardinality of  $X'$ .

## 6. W-MEASURABLE SENSITIVITY FOR FINITE MEASURE-PRESERVING TRANSFORMATIONS

In this section we restrict our attention to the case of  $T$  being an ergodic measure-preserving transformation and  $X$  being a finite measure space (so  $T$  is conservative). We shall prove our main result, that such a transformation is either W-measurably sensitive or measurably isomorphic to a Kronecker transformation. According to the results of the previous section, the same will also apply to ergodic measurably sensitive transformations.

First, we show that in the measure-preserving case, if  $d$  is a 1-Lipshitz metric with respect to  $T$ , all  $d$ -balls have the same measure.

**Lemma 6.1.** *Let  $(X, \mu, T)$  be a dynamical system with  $T$  ergodic and measure-preserving. Let  $d$  be a  $\mu$ -compatible metric that satisfies  $d(Tx, Ty) \leq d(x, y)$  for all  $x, y \in X$ .*

*Then for any  $r_0 > 0$  and any two points  $x, y \in X$ ,*

$$\mu(B_{r_0}(x)) = \mu(B_{r_0}(y)).$$

*Proof.* Let us fix a point  $x_0 \in X$  and let  $C = \mu(B_{r_0}(x_0))$ . For any  $r > 0$  let  $S_r$  denote the set  $\{x \in X \mid \mu(B_r(x)) \leq C\}$ . Our ultimate goal is to show that  $S_{r_0} = X$ . First, however, we show that for any  $r < r_0$ ,  $S_r$  covers all of  $X$ .

We begin by studying the set  $T^{-1}(S_r)$ . Suppose  $x \in X$  and  $Tx \in S_r$ . By Lemma 4.1, we have  $B_r(x) \subset T^{-1}(B_r(Tx))$ . Since  $T$  is measure-preserving,  $\mu(T^{-1}(B_r(Tx))) = \mu(B_r(Tx))$ , and therefore we have

$$\mu(B_r(x)) \leq \mu(B_r(Tx)) \leq C.$$

So  $x \in S_r$ . Since  $x$  was arbitrary,  $T^{-1}(S_r) \subset S_r$ .

Now, by Lemma 6.2 below, the set  $S_r$  is closed in the topology determined by  $d$ . In particular, as mentioned in the Introduction, this implies that  $S_r$  is measurable. Also, since  $r < r_0$ , we have  $S_r \supset B_{r_0-r}(x_0)$  (if  $y \in B_{r_0-r}(x_0)$ , then  $B_r(y) \subset B_{r_0}(x_0)$ , and  $\mu(B_r(y)) \leq C$ ), so  $S_r$  does not have measure zero. Since  $T^{-1}(S_r) \subset S_r$  and  $T$  is ergodic,  $S_r$  must have full measure. Since all non-empty  $d$ -open sets have positive measure, any closed set of full measure must be the whole space. So  $X = S_r$ .

Take any  $x \in X$ . Since we are dealing with open balls, we have

$$\mu(B_{r_0}(x)) = \sup_{r < r_0} \mu(B_r(x)).$$

Since for all  $r < r_0$ ,  $\mu(B_r(x)) \leq C$  (because  $x \in S_r$ ), we must have  $\mu(B_{r_0}(x)) \leq C$  as well. So  $x \in S_{r_0}$  and we have  $S_{r_0} = X$ .

Now, we've shown that for all  $x \in X$ ,  $\mu(B_{r_0}(x)) \leq \mu(B_{r_0}(x_0))$ . But since, in the beginning of the proof we picked  $x_0$  arbitrarily, this implies that for all  $x, y \in X$ ,  $\mu(B_{r_0}(x)) = \mu(B_{r_0}(y))$ .  $\square$

The following technical lemma was used in the above proof.

**Lemma 6.2.** *Let  $(X, \mu, T)$  be a dynamical system and  $d$  be a  $\mu$ -compatible metric on it. Let  $C > 0, r > 0$ . Then, the set  $S_r = \{x \in X \mid \mu(B_r(x)) \leq C\}$  is closed in the topology determined by  $d$ .*

*Proof.* We will show that the complement of  $S_r$ , namely that the set

$$U_r = \{x \in X \mid \mu(B_r(x)) > C\}$$

is open.

As before, we are dealing with open balls, and therefore  $\mu(B_r(x)) = \sup_{r' < r} \mu(B_{r'}(x))$ . So, since  $\mu(B_r(x)) > C$ , there must also be some  $r' < r$  such that  $\mu(B_{r'}(x)) > C$  as well.

We claim that  $B_{r-r'}(x) \subset U_r$ . Indeed, suppose  $y \in B_{r-r'}(x)$ . By the triangle inequality, we must have  $B_r(y) \supset B_{r'}(x)$ . So  $\mu(B_r(y)) > C$  and  $y \in U_r$ .

So  $U_r$  contains an open ball around  $x$ . Since  $x$  was arbitrary,  $U_r$  must be open.  $\square$

Now, we are able to show that, in the case of ergodic finite measure-preserving transformations, all 1-Lipshitz  $\mu$ -compatible metrics must in fact be isometries:

**Lemma 6.3.** *Let  $(X, \mu, T)$  be a dynamical system with  $T$  ergodic and finite measure-preserving. Let  $d$  be a  $\mu$ -compatible metric that satisfies  $d(Tx, Ty) \leq d(x, y)$  for all  $x, y \in X$ . Then  $T$  is an isometry, i.e., for any  $x, y \in X$  we have that  $d(Tx, Ty) = d(x, y)$ .*

*Proof.* Suppose that there are  $x, y \in X$  such that  $d(x, y) > d(Tx, Ty)$ . We will show that this contradicts the fact that  $T$  is measure-preserving.

Pick an  $r$  such that  $d(x, y) > r > d(Tx, Ty)$ . Notice that  $Ty \in B_r(Tx)$ . By the 1-Lipshitz condition,  $T$  is a continuous function with respect to the metric  $d$ . So the set  $T^{-1}(B_r(Tx))$  must be open, and since  $y$  is in this set, there must be a  $\delta > 0$  such that  $B_\delta(y) \subset T^{-1}(B_r(Tx))$ . Without loss of generality,  $\delta$  is small enough that  $r + \delta < d(x, y)$ .

By Lemma 4.1, we have  $B_r(x) \subset T^{-1}(B_r(Tx))$ . Now, by the triangle inequality, the sets  $B_r(x)$  and  $B_\delta(y)$  are disjoint. So we must have  $\mu(T^{-1}(B_r(Tx))) \geq \mu(B_r(x)) + \mu(B_\delta(y))$ . Since our metric is  $\mu$ -compatible,  $\mu(B_\delta(y)) > 0$ . By Lemma 6.1, we have  $\mu(B_r(x)) = \mu(B_r(Tx))$ . So we must have

$$\mu(T^{-1}(B_r(Tx))) > \mu(B_r(Tx)).$$

This contradicts our assumption that  $T$  is measure-preserving.  $\square$

Now, the following lemma establishes the connection between W-measurable sensitivity issues treated above and Kronecker transformations.

**Lemma 6.4.** *Let  $(X, \mu, T)$  be a dynamical system with  $T$  ergodic and finite measure-preserving. Let  $d$  be a  $\mu$ -compatible metric such that  $d(Tx, Ty) = d(x, y)$  for all  $x, y \in X$ . Then there exists a dynamical system  $(X_1, \mu_1, T_1)$  that is measurably isomorphic to  $(X, \mu, T)$  that is Kronecker – that is,  $X_1$  is a compact metric space and  $T_1$  is an ergodic isometry on it. Furthermore, the metric on  $X_1$  is  $\mu_1$ -compatible.*

*Proof.* We first show that  $X$  is a totally bounded space with respect to  $d$ . Let  $\varepsilon > 0$ .

Let  $C = \mu(B_{\frac{\varepsilon}{2}}(x))$  for some  $x \in X$ . Since the metric is  $\mu$ -compatible,  $C > 0$ . By Lemma 6.1, this is a constant independent of  $x$ .

Pick a largest possible collection of points  $\{x_1, \dots, x_n\}$  such that the balls  $B_{\frac{\varepsilon}{2}}(x_i)$  are all disjoint. Note that the size of any such collection will be no greater than  $\frac{\mu(X)}{C}$  (as the quantity  $\mu(\bigcup_{i=1}^n B_{\frac{\varepsilon}{2}}(x_i)) = n \cdot C$  cannot be greater than  $\mu(X)$ ). By the triangle inequality, for any point  $x \in X$ , there must be an  $i$  such that  $d(x, x_i) < \varepsilon$ , as otherwise the ball  $B_{\frac{\varepsilon}{2}}(x)$  would be

disjoint from all the  $B_{\frac{\varepsilon}{2}}(x_i)$ 's. So  $X = \bigcup_{i=1}^n B_{\varepsilon}(x_i)$ . In other words, the set  $\{x_1, \dots, x_n\}$  form an  $\varepsilon$ -net. Since  $\varepsilon$  was arbitrary,  $X$  is totally bounded.

Now, let  $(X_1, d_1)$  be the topological completion of the metric space  $(X, d)$ . It is complete and totally bounded, and therefore compact. We extend the measure  $\mu$  to  $X_1$  by defining a set  $S \subset X_1$  to be measurable if  $S \cap X$  is measurable, with  $\mu_1(S) = \mu(S \cap X)$ . Since  $T$  is an isometry, it is continuous on  $(X, d)$ , so there is a unique way to extend it to a continuous transformation  $T_1$  on  $(X_1, d_1)$ . It's easy to verify that  $T_1$  must also be an isometry with respect to  $d_1$ .

Clearly, the dynamical system  $(X_1, \mu_1, T_1)$  is measurably isomorphic to  $(X, \mu, T)$ , and  $T_1$  is an ergodic isometry on the compact metric space  $X_1$ , as desired. Finally, every  $d_1$ -ball is measurable and contains a  $d$ -ball, so the metric  $d_1$  is  $\mu_1$ -compatible.  $\square$

Finally, we are ready to state our main result, which follows directly from the two preceding lemmas.

**Theorem 1.** *Let  $(X, \mu, T)$  be a dynamical system with  $T$  ergodic, measure-preserving, and  $X$  a finite measure space. Suppose also that  $T$  is forward-measurable. Then  $T$  is either  $W$ -measurably sensitive (and therefore also measurably sensitive, according to Proposition 5.2) or  $T$  is isomorphic to a Kronecker transformation.*

*Proof.* First of all, as we discussed in Section 2, a Kronecker transformation is never  $W$ -measurably sensitive.

Now, suppose that  $T$  is not  $W$ -measurably sensitive. First of all, since  $T$  is ergodic and  $X$  has finite measure,  $T$  must be conservative (see, e.g., [Sil08]). So, by Proposition 5.1, it follows that some isomorphic dynamical system  $(X', \mu', T')$  must admit a 1-Lipshitz metric that is  $\mu'$ -compatible. Now, by Lemma 6.3 we have that  $T'$  must be an isometry with respect to that metric. Finally, by Lemma 6.4 this implies that  $T'$  is a Kronecker transformation, which concludes our proof.  $\square$

## APPENDIX A. $W$ -MEASURABLE SENSITIVITY ON MEASURABLY ISOMORPHIC DYNAMICAL SYSTEMS

In this appendix we prove that  $W$ -measurable sensitivity is invariant under measurable isomorphism. This involves a technical complication: as we will show below, it is not true in general that if two measure spaces are measurably isomorphic, a measure-compatible metric on one of the spaces can be transferred to the other space in a reasonable way, and therefore the desired result simply does not hold. However, to resolve this problem it is sufficient to restrict ourselves to dynamical systems whose underlying space is measurably isomorphic to either an interval on the real line or the whole real line, and furthermore has the same cardinality as the real line. For simplicity, throughout the rest of this appendix, by an *interval* we shall

mean either a closed interval of finite positive length or the whole real line. We have the following intermediate result.

**Lemma A.1.** *Let  $(X, \mu)$  be a measure space that is measurably isomorphic to an interval. Let  $U \subset X$  be a subset of full measure (i.e.,  $\mu(X - U) = 0$ ) and let  $d$  be a  $\mu$ -compatible metric defined on  $U$ . Suppose furthermore that the cardinalities of  $U$  and  $X$  as sets are both the same as the cardinality of the real line.*

*Then,  $d$  can be extended to a  $\mu$ -compatible metric  $d_1$  on all of  $X$  in such a way that  $d$  and  $d_1$  agree on a set of full measure.*

*Proof.* First we find some subset  $S \subset U$  that has the cardinality of the real line but has measure zero. This is possible by Lemma A.2, which is proved below.

Now, by our assumptions on cardinality of  $X$  and  $U$  there is a bijection  $\phi : (X \setminus U) \cup S \rightarrow S$ . We can extend it to a bijection  $\phi' : X \rightarrow U$  by setting

$$\phi'(x) = \begin{cases} \phi(x) & \text{if } x \in (X \setminus U) \cup S; \\ x & \text{otherwise.} \end{cases}$$

Since the set  $X \setminus U$  has measure zero and we assume that our measure is complete,  $\phi'$  is also a measurable isomorphism.

Finally, for  $x, y \in X$  we define  $d_1(x, y) = d(\phi'(x), \phi'(y))$ . Clearly, since  $d$  is a metric, so is  $d_1$ . Since every  $d_1$ -ball corresponds to a  $d$ -ball under the map  $\phi'$ , which is a measurable isomorphism,  $d_1$  is also a  $\mu$ -compatible metric.  $\square$

We used the following lemma in the above proof:

**Lemma A.2.** *Let  $I$  be an interval equipped with the Lebesgue measure  $\lambda$  and  $U \subset I$  be any subset of positive measure. Then, there exists a subset  $C \subset U$  such that  $C$  has the same cardinality as the real line and, at the same time, measure zero.*

*Furthermore, the same result holds for any set that is measurably isomorphic to a subset of  $I$  that has positive measure.*

*Proof.* We may assume that  $U$  has finite measure. Let  $M = \lambda(U)$ .

Define a map  $F : U \rightarrow [0, M]$  by  $F(x) = \lambda(U \cap (-\infty, x])$  for any  $x \in U \subset I$ . Clearly,  $F$  is a nondecreasing function on  $U$ . Since we assume that our measure is nonatomic,  $F$  is continuous and, furthermore, its values approach both endpoints of the interval  $[0, M]$  arbitrarily closely. Also, it is easy to verify that for any open interval  $(a, b) \subset [0, M]$ , we have  $\lambda(F^{-1}((a, b))) = b - a$ . Therefore  $F$  is measure-preserving.

Inside the interval  $[0, M]$ , we can find a Cantor subset  $C' \subset [0, M]$  of Lebesgue measure zero and the same cardinality as the real line. We show that the set  $C = F^{-1}(C')$  has the same properties.

Since  $F$  is continuous and its values approach both 0 and  $M$  arbitrarily closely, it must take every value in  $(0, M)$ . So the preimage of every point

of  $\mathcal{C}'$ , except possibly for 0 and  $M$ , is non-empty. The cardinality of  $\mathcal{C}$  is therefore no less than that of  $\mathcal{C}'$ , and since  $\mathcal{C} \subset I$ , it must be the same as the cardinality of the reals. The set  $\mathcal{C}$  has measure zero as  $F$  is measure-preserving. We are done proving the first statement of this lemma. The proof of second part is standard left to the reader.  $\square$

Using Lemma A.1, we can prove the invariance of W-measurable sensitivity:

**Proposition A.3.** *Suppose  $(X, \mu, T)$  is a W-measurably sensitive dynamical system such that  $X$ , as a set, has the same cardinality as the real line. Let  $(X', \mu', T')$  be a dynamical system measurably isomorphic to  $(X, \mu, T)$ . Then,  $(X', \mu', T')$  is also W-measurably sensitive.*

*Proof.* Suppose  $(X', \mu', T')$  is not W-measurably sensitive. Then, there is a  $\mu'$ -compatible metric  $d'$  on  $X'$  such that  $(X', \mu', T')$  is not W-measurably sensitive with respect to  $d'$ .

By the definition of measurable isomorphism, there must be subsets  $U \subset X$  and  $U' \subset X'$  and a measure-preserving bijection  $\phi : U \rightarrow U'$  such that  $\mu(X \setminus U) = \mu'(X' \setminus U') = 0$ , and  $\phi \circ T = T' \circ \phi$ .

We define a metric  $d$  on  $U$  by  $d(x, y) = d'(\phi(x), \phi(y))$  for  $x, y \in U$ . It is clearly  $\mu$ -compatible on  $U$ . Using our assumption on cardinality, we can apply Lemma A.1 to extend  $d$  to a  $\mu$ -compatible metric  $d_1$  defined on all of  $X$  that agrees with  $d$  almost everywhere.

Now, we want to show that  $(X, \mu, T)$  is not W-measurably sensitive with respect to  $d_1$ . Let  $\delta > 0$ . Since  $(X', \mu', T')$  is not W-measurably sensitive with respect to  $d'$ , by part 3) of Proposition 2.1, there must be an  $x' \in X'$  such that the set  $Y' = \{y \in X' \mid \forall n \geq 0, d'(T'^n x', T'^n y) < \delta/2\}$  has positive measure. Let  $Y$  be the corresponding set in  $X$ , that is  $Y = \phi^{-1}(Y' \cap U')$ . Note that  $\mu(Y) = \mu'(Y') > 0$ .

Pick any  $x \in Y$ . By the triangle inequality, for all  $y \in Y$  and all integers  $n$ , we have:

$$\begin{aligned} d_1(T^n x, T^n y) &= d'(T'^n(\phi(x)), T'^n(\phi(y))) \\ &\leq d'(x', T'^n(\phi(x))) + d'(x', T'^n(\phi(y))) \\ &\leq \delta. \end{aligned}$$

Since  $Y$  has positive measure,  $(X, \mu, T)$  cannot be W-measurably sensitive. So the property of W-measurable sensitivity is preserved under measurable isomorphism.  $\square$

Now that we showed the desired result, we note that the restriction on the cardinality of the space is not a very strong one, as any space measurably isomorphic to an interval of non-zero length can be expressed as a union of a set of measure zero and a set of cardinality the same as that of the reals.

Nevertheless, this restriction is necessary for W-measurable sensitivity to be preserved under measurable isomorphisms. The reason for this stems from the following result:

**Proposition A.4.** *Let  $(X, \mu)$  be a measure space and  $d$  be a  $\mu$ -compatible metric on it. Then,  $X$  cannot have cardinality greater than that of the reals.*

*Proof.* As  $d$  is a  $\mu$ -compatible metric on  $(X, \mu)$ ,  $X$  must be a separable metric space under  $d$  [JKL<sup>+</sup>08]. If  $F$  is a countable dense subset of  $X$ , every element of  $X$  corresponds to a Cauchy sequence of elements of  $F$ . So  $X$  has at most the cardinality of the reals.  $\square$

According to this proposition, if  $X$  has too large a cardinality, even if it is measurably isomorphic to the unit interval, it will admit no measure-compatible metric. This would imply, for instance, that any transformation  $T$  on  $X$  would vacuously be W-measurably sensitive.

In particular, allowing such spaces means that W-measurable sensitivity is no longer invariant under measurable isomorphism. For instance, let  $I$  be the unit interval and  $S$  be a set with cardinality larger than that of the reals. Let  $X$  be the disjoint union of  $I$  and  $S$ , with a measure defined on it so that any subset of  $S$  has zero measure, and any measurable subset of  $I$  has the Lebesgue measure. Let  $T$  be the identity transformation on  $X$ . By the above argument, the dynamical system  $(X, T)$  would be W-measurable sensitive, even though the measurably isomorphic system  $(I, T)$  is clearly not.

There are options of avoiding this problem other than restricting the cardinality of our spaces. One such way would be to loosen the definition of a  $\mu$ -compatible metric. We can say that a measure space  $(X, \mu)$  possesses a  $\mu$ -compatible metric  $d$  if  $d$  is a metric defined on a subset  $U \subset X$  of full measure, and the balls of  $d$  have positive  $\mu$ -measure. In that case, Lemma A.1 would be tautologically true for any measure space  $X$ , and therefore the proof of Proposition A.3 would also hold for arbitrary measurably isomorphic dynamical systems.

Yet another way of avoiding the same problem would be to follow [JKL<sup>+</sup>08] and define a system  $(X, \mu, T)$  to be W-measurably sensitive if for all measurably isomorphic systems  $(X', \mu', T')$  and all  $\mu'$ -compatible metrics  $d'$  on  $X'$ , the system  $(X', \mu', T')$  is W-measurably sensitive with respect to  $d'$ .

## APPENDIX B. PAIRWISE SENSITIVITY

In their paper [CJ05], Cadre and Jacob introduce the notion of *pairwise sensitivity*, which they define as follows. They only consider finite measure-preserving transformations, so we will restrict to them in this appendix.

**Definition B.1.** *Let  $(X, \mu)$  be a Lebesgue probability space and let us fix a metric  $d$  on  $X$ .*

*An endomorphism  $T$  is said to be pairwise sensitive (with respect to initial conditions) if there exists  $\delta > 0$  — a sensitivity constant — such that for  $\mu^{\otimes 2}$ -a.e.  $(x, y) \in X \times X$ , one can find  $n \geq 0$  with  $d(T^n x, T^n y) \geq \delta$ .*

Since this concept depends on the choice of a metric  $d$ , we will often refer to  $T$  as being *pairwise sensitive with respect to  $d$* .

Cadre and Jacob prove that weakly mixing finite measure-preserving transformations are pairwise sensitive, and that a certain entropy condition implies pairwise sensitivity for ergodic transformation.

This notion is very closely related to the notion of W-measurable sensitivity, as the following proposition shows.

**Proposition B.1.** *Let  $(X, \mu, T)$  be a dynamical system and  $d$  be a  $\mu$ -compatible metric on  $X$ . Then  $T$  is pairwise sensitive with respect to  $d$  if and only if it is W-measurably sensitive with respect to  $d$ .*

*Proof.* First suppose that the system is W-measurable sensitive with respect to  $d$ . Then, there is a  $\delta > 0$  such that for every  $x \in X$  the set

$$Y_x = \{y \in X \mid \exists n \text{ such that } d(T^n x, T^n y) > \delta\}$$

has full measure. By Fubini's theorem (for the version we use, see [EG92]), the set  $Y = \{(x, y) \in X \times X \mid \exists n \text{ such that } d(T^n x, T^n y) > \delta\}$  must have full  $\mu^{\otimes 2}$ -measure in  $X \times X$ . So  $T$  is pairwise sensitive with respect to  $d$ .

Now, suppose that the system is pairwise sensitive with respect to a  $\mu$ -compatible metric  $d$ . That is, there is a  $\delta > 0$  such that the set  $Y$ , defined as before, has full  $\mu^{\otimes 2}$ -measure in  $X \times X$ .

Take any  $x \in X$ . We claim that for almost every  $y \in X$ , there is an  $n$  such that  $d(T^n x, T^n y) > \delta/2$ . Once we have this claim, Proposition 2.1 implies that  $T$  is W-measurably sensitive with respect to  $d$ .

To prove the above claim, we need to show that the set  $S_x = \{y \in X \mid \forall n, d(T^n x, T^n y) \leq \delta/2\}$  has measure zero. Take any  $y_1, y_2 \in S_x$ . By the triangle inequality, for all  $n$  we have  $d(T^n y_1, T^n y_2) \leq \delta$ . So the pair  $(y_1, y_2)$  does not belong to the set  $Y \subset X \times X$ . In other words, the Cartesian product  $S_x \times S_x$  lies wholly inside the  $\mu^{\otimes 2}$ -measure-zero set  $(X \times X) \setminus Y$ . Again by Fubini, this is only possible if the set  $S_x$  is measurable and has  $\mu$ -measure zero.  $\square$

With this in mind, our Theorem 5.1 implies the following theorem concerning pairwise sensitivity.

**Theorem 2.** *Let  $(X, \mu, T)$  be a nonatomic ergodic finite measure-preserving dynamical system such that  $T$  is forward measurable. Suppose further that this dynamical system is not measurably isomorphic to a Kronecker transformation. Then, for any  $\mu$ -compatible metric  $d$ ,  $T$  is pairwise sensitive with respect to  $d$ .*

Note that many of the additional assumptions we need to make do not significantly narrow down the applicability of this result. Cadre and Jacob also work in the finite measure-preserving setting, and their main result also applies only to ergodic transformations. Just like we showed with W-measurable sensitivity in the Remark in Section 2, if a measure space has points of positive measure, no transformation on it can exhibit pairwise sensitivity with respect to any metric.

We do need to assume that the metric  $d$  is  $\mu$ -compatible. However, while Cadre and Jacob never specify any restrictions on their metric, they also tacitly use several very similar properties. For example, they extensively use the notion of the *support* of a measure (i.e., the complement of the largest open set of zero measure), which is not well-defined without the assumptions that open and closed sets are measurable, and that the space is separable (if the space were not separable, the union of all open sets of measure zero may have positive measure even if measurable). Together, these two properties are almost sufficient to force the metric to be  $\mu$ -compatible, as the following proposition shows.

**Proposition B.2.** *A metric  $d$  on a measure space  $(X, \mu)$  is  $\mu$ -compatible if and only if the following three conditions are satisfied:*

- (1) *Every  $d$ -ball is  $\mu$ -measurable.*
- (2) *The space  $X$  is separable under  $d$ .*
- (3) *The support of  $\mu$  is the whole of  $X$ .*

*Proof.* The fact that if  $d$  is  $\mu$ -compatible then  $X$  is separable under  $d$  is shown by [JKL<sup>+</sup>08]. The other two properties are obvious.

Now, suppose that  $d$  satisfies all of the first two properties. Then, the notion of the support is well-defined. Clearly, the support of the measure is the whole space if and only if every non-empty open set has positive measure, i.e., if  $d$  is  $\mu$ -compatible.  $\square$

According to this Proposition B.2, to go from  $\mu$ -compatible metrics to the metrics Cadre and Jacob use, we only need to require that the support of the measure is the whole space. This can always be achieved by removing a set of measure zero from the space.

So the only significant additional assumptions we need to make is that  $T$  is forward-measurable. With this assumption, Theorem 2 sharpens the results of [CJ05].

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